21 Laplace's Equation and Harmonic Functions

21.1 Introductory Remarks on the Laplacian operator

Given a domain $\Omega \subset \mathbb{R}^d$, then

$$\nabla^2 u = \mathrm{div}(\mathrm{grad}\ u) = 0 \quad \text{in } \Omega \tag{1}$$

is **Laplace's equation** defined in Ω. If $d = 2$, in cartesian coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

or, in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\frac{\partial u}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} .$$

If $d = 3$, in cartesian coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

while in cylindrical coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\frac{\partial u}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

and in spherical coordinates

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left\{ \rho^2\frac{\partial u}{\partial \rho}\right\} + \frac{1}{\rho^2}\left\{ \frac{\partial^2 u}{\partial \phi^2} + \cot\phi \frac{\partial u}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right\} .$$

We deal with problems involving most of these cases within these Notes.

Remark: There is not a universally accepted angle notion for the Laplacian in spherical coordinates. See Figure 1 for what θ, ϕ mean here. (In the rest of these Notes we will use the notation r for the radial distance from the origin, no matter what the dimension.)

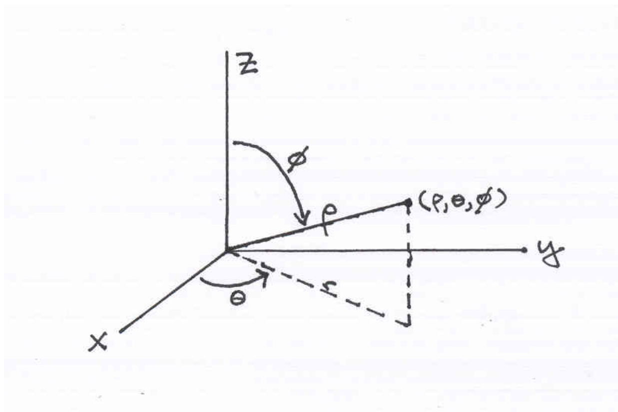

Figure 1: Coordinate angle definitions that will be used for spherical coordinates in these Notes.

Remark: In the math literature the Laplacian is more commonly written with the symbol Δ; that is, Laplace's equation becomes $\Delta u = 0$. For these Notes we write the equation as is done in equation (1) above. The non-homogeneous version of Laplace's equation, namely

$$\nabla^2 u = f(\mathbf{x}) \tag{2}$$

is called **Poisson's equation**. Another important equation that comes up in studying electromagnetic waves is **Helmholtz's equation**:

$$\nabla^2 u + k^2 u = 0 \qquad k^2 \text{ is a real, positive parameter} \tag{3}$$

Again, Poisson's equation is a non-homogeneous Laplace's equation; Helmholtz's equation is not.

Laplace introduced the notion of a potential as the gradient of forces on a celestial body in 1785, and this potential turned out to satisfy Laplace's equation. Then other applications involving Laplaces's equation came along, including steady state heat flow (Fourier, 1822), theory of magnetism (Gauss and Weber, 1839), electric field theory (Thomson, 1847), complex analysis (Cauchy, 1825, Riemann, 1851), irrotational fluid motion in 2D (Helmholtz, 1858). For the fluid case let $\mathbf{u} = (u, v)$ be the fluid velocity vector. Incompressibility gives the condition $div(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0$, and irrotationality gives the condition $curl(\mathbf{u}) = \nabla \times \mathbf{u} = \mathbf{0}$; this allows us to write $\mathbf{u} = -grad\ \phi$, where ϕ is called a *velocity potential*. Thus, $0 = div(\mathbf{u}) = -div(grad\ \phi)$, so the velocity potential satisfies Laplace's equation in the fluid domain.

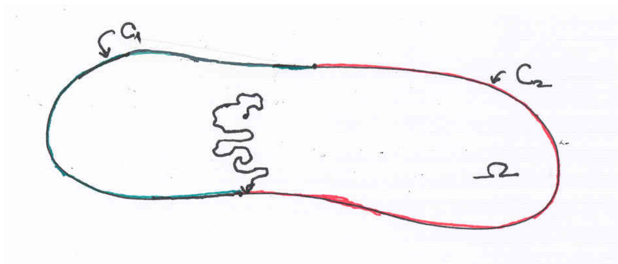

Figure 2: Domain with boundary partitioned into two disjoint pieces for the probability example.

Complex variable theory is the study of analytic functions $f = f(z)$ of the complex variable $z = x + iy$ (so f has a convergent power series in z). We write $f(z) = u(z) + iv(z)$, where u and v are real valued functions, and thinking of them as functions of x and y, they satisfy the *Cauchy-Riemann equations* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Note that this gives

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \text{ ; that is, } \nabla^2 u = 0 \text{ .}$$

Similarly, $\nabla^2 v = 0$.

Definition: If u is twice differentiable in each variable in domain Ω, and $\nabla^2 u = 0$ in Ω, then u is called **harmonic** in Ω (or is a **harmonic function in** Ω).

Hence, in the above example, u and v are harmonic real and imaginary parts of the analytic function f. Note that any constant is harmonic (everywhere) and that if u is harmonic in a domain, so is any constant multiple of u.

For another example suppose Ω is a bounded domain in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and suppose $\partial\Omega = C_1 \cup C_2$, C_1, C_2 are nonempty and non-intersecting. That is, the boundary of Ω is made up of two disjoint nonempty pieces (see Figure 2). For the sake of argument we will consider the 3D space case and define $u(x, y, z) =$ probability that a particle that begins at $(x, y, z) \in \Omega$ and moves as a random Brownian motion particle, will stop at a point of C_2. Then it turns out that u satisfies the problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = 1 & \text{on } C_2 \\ u = 0 & \text{on } C_1 \end{cases}$$

Here is another important example. Let $\mathbf{E}$ be the electric field vector and $\mathbf{H}$ be the magnetic field vector. The **Maxwell's equations** from electromagnetic field theory can be written as

$$\nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}\mathbf{H} \tag{4}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{5}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial}{\partial t}\mathbf{E} + \mathbf{J} \tag{6}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon \tag{7}$$

where μ is the magnetic permeability, ρ is charge density, ϵ is the dielectric constant, and $\mathbf{J}$ is a (generally specified) current density. If the medium is homogeneous, all the parameters are constant. Equation (6) is sometimes called *Faraday's law*, and (7) is sometimes called *Coulomb's law*. In electrostatics, the t-derivatives are zero, so from (4), $\nabla \times \mathbf{E} = 0$ (i.e. $\mathbf{E}$ is an irrotational field). Therefore, there exists an electrical potential ϕ such that $\mathbf{E} = -\nabla\phi$. So, $div\ \mathbf{E} = -\ div\ (\nabla\phi) = -\nabla^2\phi = \rho/\epsilon$, that is, the electric potential satisfies Poisson's equation. If we are in a vacuum and consider $\rho = 0$, then ϕ satisfies Laplace's equation. If $\mathbf{J} = 0$, then $\mathbf{H}$ can be associated with a magnetic potential, ψ, and ψ satisfies Laplace's equation because of (5).

In this section we mostly are concerned with Laplace's equation in 2D spatial domains, and construct solutions in some special cases. Again we do this via separation of variables method. This is mostly a change of notation rather than any new ideas. We restrict our problems to rectangular (cartesian) coordinates and polar coordinates, mainly because of computational "simplicity" since problems in 3 space variables can get really long and messy without introducing any new technical ideas. (We do one case in spherical coordinates in Appendix H.)

21.2 Laplace's Equation in a Rectangle

Consider

$$\nabla^2 u = 0 \quad \text{in } \Omega = \{(x,y) : 0 < x < K, 0 < y < L\}$$

along with Dirichlet boundary conditions as given in Figure 3. (There is nothing special about using Dirichlet b.c.s on each of the four sides; we could

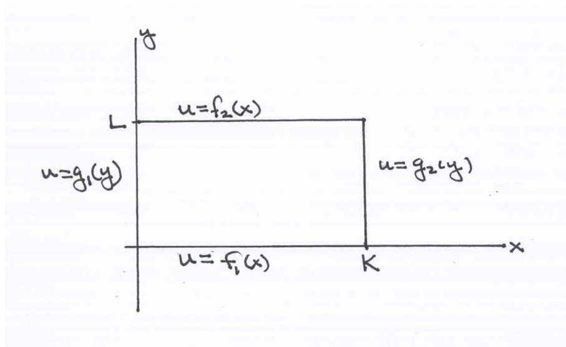

Figure 3: Rectangular domain in $\mathbb{R}^2$ with Dirichlet data on all sides.

have picked Neumann or Robin conditions for any, or all, of the sides. So we can consider this problem is one of 12 possible problems to discuss.) By linearity we view the solution u as the sum of four functions (if we have non-homogeneous b.c.s on all four sides), $u = u^1 + u^2 + u^3 + u^4$, where u^j satisfies Laplace's equation in Ω and satisfies one non-homogeneous b.c. while having homogeneous b.c.s on the other 3 sides. For example, assume u^2 be the solution the the problem pictured in Figure 4. By separation of variables method, let $u^2 = X(x)Y(y)$ and substitute this into the equation. Then

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \;\Rightarrow\; \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

so

$$\frac{d^2Y}{dy^2} + \lambda Y = 0\ ,\ 0 < y < L\ ,\ Y(0) = Y(L) = 0\ .$$

The eigenvalues are just $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$, with associated eigenfunctions $Y_n(y) = \sin(\frac{n\pi y}{L})$. Thus, $\frac{d^2X}{dx^2} - \lambda X = \frac{d^2X}{dx^2} - (n\pi/L)^2 X = 0$, $0 < x < K$, $X(0) = 0$. This gives $X(x) = X_n(x) = \sinh(\frac{n\pi x}{L})$, up to a multiplicative constant. Therefore, each "mode" of u is made up of $\sinh(\frac{n\pi x}{L})\sin(\frac{n\pi y}{L})$, and adding up all contributions gives

$$u^2(x,y) = \sum_{n=1}^{\infty} b_n \sinh(\frac{n\pi x}{L})\sin(\frac{n\pi y}{L})$$

Lastly, we consider the one non-homogeneous b.c., i.e.

$$g_2(y) = u^2(K,y) = \sum_{n=1}^{\infty} b_n \sinh(\frac{n\pi K}{L})\sin(\frac{n\pi y}{L})\ .$$

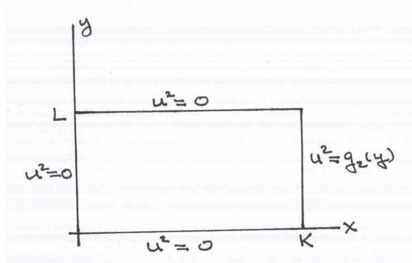

Figure 4: Rectangular domain in $\mathbb{R}^2$ with nonzero data only on one side.

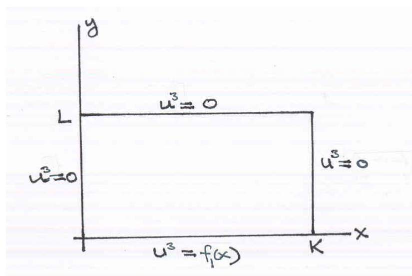

Figure 5: Rectangular domain in $\mathbb{R}^2$ with nonzero data on a different side.

This is just the Fourier sine series for $g_2(y)$ on $(0, L)$ with Fourier coefficient $b_n \sinh(\frac{n\pi K}{L})$. Therefore,

$$b_n = \frac{2}{L \sinh(\frac{n\pi K}{L})} \int_0^L g_2(y) \sin(\frac{n\pi y}{L}) dy \ .$$

The other three problems are done the same way. The ODE boundary-value problem with homogeneous boundary conditions at both ends of their interval determines the EVP. So, for the case in Figure 5, it is the x-variable problem ($u^3 = X(x)Y(y)$) that determines the eigenvalues and eigenfunctions. In this case, $\lambda_n = (\frac{n\pi}{K})^2, X_n(x) = \sin(\frac{n\pi x}{k})$, so that $Y'' - \lambda_n Y = 0, Y(L) = 0$. For convenience, you can write $Y(y) = Y_n(y) = \sinh(\frac{n\pi}{K}(L - y))$, rather than a linear combination of $\sinh(n\pi y/L), \cosh(n\pi y/l)$. Then

$$u^3(x, y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi}{K}(L - y)) \sin(\frac{n\pi x}{K}) \ \text{ with}$$

$$a_n = \frac{2}{K \sinh(\frac{n\pi L}{K})} \int_0^K f_1(x) \sin(\frac{n\pi x}{K}) dx \ .$$

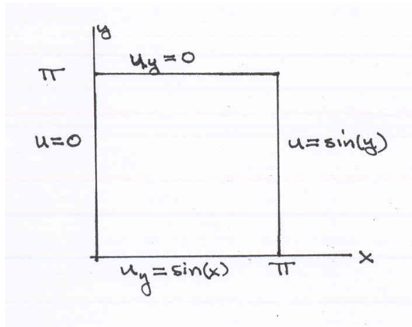

Figure 6: Square domain and boundary conditions for the Exercise.

Try the following problem:

Exercise: Let $\nabla^2 u = 0$ in the square $\Omega = \{(x,y) : 0 < x, y < \pi\}$ with the boundary data given as in Figure 6. Find $u(x,y)$.

21.3 Laplace's Equation in a Disk: Poisson's Formula

Consider

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(x,y) : x^2 + y^2 < a^2\} = \{(r,\theta) : 0 \le r < a, 0 \le \theta < 2\pi\} \\ u(a,\theta) = f(\theta) & 0 \le \theta < 2\pi \end{cases}$$

Now, in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \ ,$$

so let $u = \Theta(\theta)\phi(r)$. Then, by separation of variables,

$$\frac{r\frac{d}{dr}(r\frac{d\phi}{dr})}{\phi} = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda \ \Rightarrow$$
$$\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0 \ \ 0 \le \theta < 2\pi \ \ \Rightarrow$$
$$\Theta(\theta) = a_n \cos(\sqrt{\lambda}\theta) + b_n \sin(\sqrt{\lambda}\theta) \ .$$

But Θ must be 2π-periodic, which forces $\lambda = \lambda_n = n^2$, for $n = 0, 1, 2, \ldots$, so $\Theta = \Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$. Note that $\lambda_0 = 0$ is an eigenvalue.

Also,

$$r^2\frac{d^2\phi}{dr^2}+r\frac{d\phi}{dr}-n^2\phi=0\ \ . \tag{8}$$

If $n=0$, then

$$r^2\frac{d^2\phi}{dr^2}+r\frac{d\phi}{dr}=r\frac{d}{dr}(r\frac{d\phi}{dr})=0\Rightarrow\phi=\phi_0=c\ln(r)+d\ .$$

Since u, and hence ϕ, must be bounded at $r=0$, then $c=0$; i.e. $\phi_0=$ constant. If $n>0$, then (8) is a Cauchy-Euler equation. Therefore, let $\phi=r^\alpha$; then the characteristic equation for (8) when this is substituted in becomes $\alpha(\alpha-1)+\alpha-n^2=\alpha^2-n^2=0$, so $\alpha=\pm n$. The general solution, for $n>0$, is now

$$\phi=\phi_n(r)=\alpha_n r^n+\beta_n r^{-n}\ \ . \tag{9}$$

Again, since we require boundedness at $r=0$, $\beta_n=0$ for all $n\geq 1$. In summary, $\phi_n(r)$ is a constant for $n=0$ (so is $\Theta=\Theta_0(\theta)$) and is a constant times r^n for $n>0$. Combining this with $\Theta_n(\theta)$ gives

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\{a_n\cos(n\theta)+b_n\sin(n\theta)\}\ . \tag{10}$$

Letting $r\to a$ gives

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a^n\{a_n\cos(n\theta)+b_n\sin(n\theta)\}\ .$$

This is the full Fourier series for $f(\theta)$. Using the orthogonality of the set $\{1,\cos(n\theta),\sin(n\theta)\}_{n\geq 1}$ on $[0,2\pi]$, we have

$$\frac{a_0}{2}=\frac{1}{2\pi}\int_0^{2\pi}f(\psi)d\psi=\text{ average of } f \text{ on } [0,2\pi]\ , \tag{11}$$

and

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix}=\frac{1}{\pi a^n}\int_0^{2\pi}f(\psi)\begin{pmatrix}\cos(n\psi)\\ \sin(n\psi)\end{pmatrix}d\psi\ \ . \tag{12}$$

This finishes the description of the solution (10) to our potential equation in the disk. But let us push the result a bit further by substituting (11),(12) into (10):

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)d\psi +$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi} f(\psi)\{\cos(n\psi)\cos(n\theta) + \sin(\psi)\sin(\theta)\}d\psi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\left\{1 + 2\sum_1^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\psi)\right\} d\psi$$

Consider now the expression in the brackets with $\xi := \theta - \psi$:

$$1 + 2\sum_1^{\infty}\left(\frac{r}{a}\right)^n \cos(n\xi) = 1 + \sum_1^{\infty}\left(\frac{r}{a}\right)^n e^{in\xi} + \sum_1^{\infty}\left(\frac{r}{a}\right)^n e^{-in\xi} .$$

Recall the geometric series $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ if $|z| < 1$. Since $r < a$, $|z| = |\frac{r}{a}e^{\pm i\xi}| = |\frac{r}{a}| < 1$, so

$$\begin{aligned}
&1 + \sum_1^{\infty}\left(\frac{r}{a}e^{i\xi}\right)^n + \sum_1^{\infty}\left(\frac{r}{a}e^{-i\xi}\right)^n \\
&= 1 + \frac{(r/a)e^{i\xi}}{1-(r/a)e^{i\xi}} + \frac{(r/a)e^{-i\xi}}{1-(r/a)e^{-i\xi}} \\
&= 1 + \frac{re^{i\xi}}{a - re^{i\xi}} + \frac{re^{-i\xi}}{a - re^{-i\xi}} \\
&= 1 + \frac{re^{i\xi}(a - re^{-i\xi}) + re^{-i\xi}(a - re^{i\xi})}{a^2 - ar(e^{i\xi} + e^{-i\xi}) + r^2} \\
&= \frac{a^2 - r^2}{a^2 - 2ar\cos(\xi) + r^2} \quad .
\end{aligned}$$

Hence

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(\psi)\, d\psi}{a^2 - 2ar\cos(\theta-\psi) + r^2} . \qquad (13)$$

This is **Poisson's Formula** for any harmonic function inside a circle of radius $a > 0$ (centered at the origin), with boundary values given by $f(\psi)$.

Theorem: Let $f(\theta)$ be any continuous function on the boundary of the disk $\{r = a\}$. Then (13) provides the only harmonic function in $\Omega = \{r < a\}$

for which for any $\mathbf{x}_0 = (a, \theta_0) \in \partial\Omega = \{r = a\}$, $lim_{\mathbf{x}\to\mathbf{x}_0} u(\mathbf{x}) = f(\mathbf{x}_0)$ from inside Ω.

This means that u is at least continuous in Ω and its boundary, and twice continuously differentiable inside Ω. In actuality, u is infinitely differentiable inside Ω.

Exercise: Derive the series solution and analogue to Poisson's formula for the **exterior problem**:

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : r > a, 0 \le \theta < 2\pi\} \\ u(a,\theta) = f(\theta) & 0 \le \theta < 2\pi \end{cases}$$

21.4 Some Comments and Consequences

1. Let $\mathbf{x} = (r\cos(\theta), r\sin(\theta)), \mathbf{x}' = (a\cos(\psi), a\sin(\psi))$, then $|\mathbf{x} - \mathbf{x}'|^2 = (r\cos(\theta) - a\cos(\psi))^2 + (r\sin(\theta) - a\sin(\psi))^2 = r^2 - 2ar\cos(\theta - \psi) + a^2$, so Poisson's formula (13) can be rewritten as

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} ds' \ , \tag{14}$$

   where arclength on the circle is $ds' = a\ d\theta$.

2. **Mean Value Property:** Let $\mathbf{x} = \mathbf{0}$ ($r = 0$) in (13) (or (14)):

$$u(0,\theta) = u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi) d\psi = \text{ average of } f$$

   That is, the value of a harmonic function u at the center of a disk equals the average of u on its circumference (if the disk is contained in a region where u is harmonic). This also means $min_{r=a} u \le u(\mathbf{0}) \le max_{r=a} u$ because of the mean value property, and equality holds for one, hence both, sides of the inequality only if $u =$ constant. It also follows that if *u is harmonic in domain Ω, and if C is any circle contained in Ω with center located at $\boldsymbol{x} \in \Omega$, then $u(\boldsymbol{x})$ = average of values of u around C.* This leads to

3. **Maximum Principle:** For any bounded, connected domain Ω, let u be continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$, harmonic in Ω. Then the maximum (and minimum) of u is attained on $\partial\Omega$, and nowhere else (unless $u =$ constant).

   Suppose u attained a maximum at some $\mathbf{x}^* \in \bar{\Omega}$, that is $u(\mathbf{x}) \leq u(\mathbf{x}^*) = M$ for all $\mathbf{x} \in \Omega$. Then the claim is that $\mathbf{x}^* \notin \Omega$, unless $u =$ constant. But if $\mathbf{x}^* \in \Omega$ it is the center for a disk C contained entirely in Ω. By the mean value property, $u(\mathbf{x}^*)$ is the average of the values of u around the boundary of the C, and since the average can not be greater than M, then $M = u(\mathbf{x}^*) =$ average of u on the circle $\leq M$, which implies $u \equiv M$ along the whole circumference of C. This makes $u \equiv M$ in the whole disk C. Repeating this argument for another overlapping disk forces u to be M in the union of the disks. Continue to move to new disks, one obtain $u \equiv M$ throughout Ω; hence u is a constant in its domain. This argument can be repeated for $-u$ since it would also be harmonic in Ω, and so the minimum of u is attained on $\partial\Omega$ (unless u is a constant).

4. **Smoothness:** Let u be harmonic in $\Omega \subset \mathbb{R}^2$. Then $u(\mathbf{x})$ possesses partial derivatives of all orders in Ω.

   Remark: We mentioned earlier that this smoothness property and the maximum principle property holds for the heat equation. Actually, in that case, they are a consequence of the properties holding for the Laplacian (but they absolutely do not hold for the wave equation).

5. **Harnack's inequality:** Let u be harmonic and non-negative in the disk $\Omega = \{|\mathbf{x}| < a\}$. Then, for any $\mathbf{x} = (r, \theta) \in \Omega$,

$$\frac{a-r}{a+r} u(\mathbf{0}) \leq u(\mathbf{x}) \leq \frac{a+r}{a-r} u(\mathbf{0}) \ .$$

   Exercise: This is just a consequence of Poisson's formula and the mean value property. Try to prove it.

6. **Liouville's Theorem:** A function that is harmonic in the whole plane, and bounded either from above, or below, is a constant.

Suppose u is harmonic in $\mathbb{R}^2$ and $u(\mathbf{x}) < M$ for all $\mathbf{x} \in \mathbb{R}^2$. Then $M - u(\mathbf{x})$ is harmonic and non-negative in $\mathbb{R}^2$. Apply Harnack's inequality for a disk centered at the origin and having radius a:

$$\frac{a-r}{a+r}(M - u(\mathbf{0})) \leq M - u(\mathbf{x}) \leq \frac{a+r}{a-r}(M - u(\mathbf{0})) \ .$$

Now let $a \to \infty$. this gives $M - u(\mathbf{0}) \leq M - u(\mathbf{x}) \leq M - u(\mathbf{0})$; that is, $u(\mathbf{x}) \equiv u(\mathbf{0}) \quad \Rightarrow \quad u$ is a constant. For the case where $u(\mathbf{x}) > m$ for all $\mathbf{x} \in \mathbb{R}^2$, use the same argument for $u(\mathbf{x}) - m$.

7. **Dirichlet's Principle:** A general principle in physics is that a system prefers going into a state of lowest energy (the 'ground state'). Dirichlet's principle codifies this mathematically for Laplace's equation on a bounded domain (in $\mathbb{R}^n$) with specific Dirichlet boundary conditions, and states that of all smooth functions defined on the domain and satisfying the boundary conditions, it is the harmonic function on the domain that has minimum potential energy. The Dirichlet principle is discussed in Appendix I.

21.5 Laplace's Equation on a Wedge

Consider

$$\left.\begin{array}{ll} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 < r < a, 0 < \theta < \theta_0\} \\ u(a,\theta) = f(\theta) & 0 < \theta < \theta_0 \\ u(r,0) = u(r,\theta_0) = 0 & 0 < r < a \end{array}\right\} \qquad (15)$$

As before, let $u(r,\theta) = \phi(r)\Theta(\theta)$. Then

$$\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0 \ , \ \ 0 < \theta < \theta_0 \ , \ \Theta(0) = \Theta(\theta_0) = 0 \ .$$

and

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \lambda\phi = 0 \ , \ \ 0 < r < a \ .$$

Now $\Theta(\theta) = a\cos(\sqrt{\lambda}\theta) + b\sin(\sqrt{\lambda}\theta)$; with $\Theta(0) = a = 0$, $\Theta(\theta_0) = b\sin(\sqrt{\lambda}\theta_0) = 0$, then we have $\lambda = \lambda_n = \left(\frac{n\pi}{\theta_0}\right)^2$, $n = 1, 2, \ldots$, and so $\Theta(\theta) = \Theta_n(\theta) = \sin(n\pi\theta/\theta_0)$. Also,

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} - \left(\frac{n\pi}{\theta_0}\right)^2 \phi = 0 \ .$$

Again, this is a Cauchy-Euler equation, with general solution $\phi(r) = \phi_n(r) = ar^{n\pi/\theta_0} + br^{-n\pi/\theta_0}$ because the characteristic polynomial for the equation is $\alpha^2 - \lambda_n = 0$, so a fundamental set of solutions is $r^{\sqrt{\lambda_n}}, r^{-\sqrt{\lambda_n}}$. But we want boundedness of u, hence ϕ as $r \to 0$ within the wedge, so we need $b = 0$. Therefore,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\theta_0} \sin(n\pi\theta/\theta_0) \ .$$

Letting $r \to a$ we obtain the Fourier sine series for f:

$$f(\theta) = \sum_{n=1}^{\infty} A_n a^{n\pi/\theta_0} \sin(n\pi\theta/\theta_0) \quad \Rightarrow$$
$$A_n = \frac{2a^{-n\pi/\theta_0}}{\theta_0} \int_0^{\theta_0} f(\theta) \sin(n\pi\theta/\theta_0) d\theta \ .$$

Remark: On the more general wedge problem
Consider the problem

$$\left.\begin{array}{ll} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 < r < a, 0 < \theta < \theta_0\} \\ u(a,\theta) = f(\theta) & 0 < \theta < \theta_0 \\ u(r,0) = A(r) & \\ u(r,\theta_0) = B(r) & 0 < r < a \end{array}\right\} \tag{16}$$

It is natural to break this problem up into two pieces, one being problem (15) with solution $u^{(1)}$ given above, and one being

$$\left.\begin{array}{ll} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 < r < a, 0 < \theta < \theta_0\} \\ u(a,\theta) = 0 & 0 < \theta < \theta_0 \\ u(r,0) = A(r) & \\ u(r,\theta_0) = B(r) & 0 < r < a \end{array}\right\} \tag{17}$$

with solution $u^{(2)}$. Then $u = u^{(1)} + u^{(2)}$. Unfortunately, problem (17) takes more advanced techniques to solve than what we are willing to do at this subject level.

Remark: In three space dimensions we would be interested in Laplace's equation in the ball. This is discussed in Appendix H, where another historically important equation and special functions are introduced, namely Legendre's equation and Legendre polynomials.

Summary: You need to know how to compute eigenfunction expansions for harmonic functions on specific domains, like rectangles, disks, and wedges. You need to know properties of harmonic functions as given in this section.

Exercises:

1. Let $\nabla^2 u = 0$ in the square $\Omega = \{(x, y) : 0 < x, y < \pi\}$, with $u_x(\pi, y) + au(\pi, y) = G$, and $u = 0$ on the other three sides of the square. Assume G and a are constants, with $a \neq -1$. Determine the eigenfunction expansion for the solution $u(x, y)$.
(Answer: $u(x, y) = \frac{4G}{\pi} \sum_{n=odd}^{\infty} \frac{\sinh(nx)}{n\cosh(n\pi)+a\sinh(n\pi)} \sin(ny)$.)

2. Consider $\nabla^2 u = 0$ in the unit disk $\Omega = \{(r, \theta) : r < 1, 0 < \theta \leq 2\pi\}$, with $\frac{\partial u}{\partial r}(1, \theta) = g(\theta)$, where $\int_0^{2\pi} g(\theta)\, d\theta = 0$. Find a formal solution.

3. Let $\nabla^2 u = 0$ in the rectangle $\Omega = \{(x, y) : 0 < x < K, 0 < y < L\}$, with boundary conditions $u_x(0, y) = 0, u_y(x, 0) = u_y(x, L) = 0$, and $u(K, y) = A =$ constant. Determine the harmonic solution in Ω.
(Answer: $u(x, y) \equiv A$.)

4. Consider $\nabla^2 u = c$ in the unit disk $\Omega = \{(r, \theta) : r < 1, 0 < \theta \leq 2\pi\}$, where c is a constant, and $\partial u/\partial r = 1$ on the boundary $\partial\Omega = \{r = 1\}$. Determine all solutions. (Hint: boundary condition is independent of the angle θ, so so is the solution.)

5. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \alpha\frac{\partial u}{\partial z} = 0 \quad |x| < l\ ,\ z > 0\ ,\ \alpha > 0 \text{ is a constant}$$

Assume u is to remain finite as $z \to \infty$, and has the form

$$u(x, z) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(z) \cos(\frac{n\pi x}{l})\ .$$

Show that

$$u(x, z) = a_0/2 + \sum_{n=1}^{\infty} a_n e^{\beta_n z} \cos(\frac{n\pi x}{l})$$

where the $a_n's, \beta_n's$ are constants to be determined.

Now assume the boundary condition

$$-\frac{\partial u}{\partial z}(x,0) + \alpha u(x,0) = RH(\omega - |x|)$$

where $R > 0$ is constant, ω is some fixed constant in $(0, l)$, and $H(\cdot)$ is the Heaviside function. Find the solution $u(x, z)$[1].

6. Consider the Neumann problem

$$\begin{cases} \nabla^2 u = f(x,y) & \text{in } \Omega \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where ν is the unit outward pointing vector defined on the boundary of domain Ω.

(a) What can we add to any solution to get another solution?

(b) Use the divergence theorem and the PDE to show that $\int_\Omega f(x,y)dxdy = 0$ is a necessary condition for the Neumann problem to have a solution.

(c) Can you give a physical interpretation of (a) and (b) for heat flow or general diffusion?

(If the boundary is completely insulated, then the non-homogeneity has average (over Ω) zero, meaning the internal sources and sinks must "balance.")

[1]This problem is associated with a certain irrigation problem.